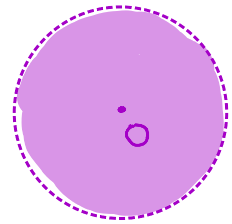
Suprising application of the maximum modulus principle, related to section 5.2 and the *hyperbolic plane* in geometry. This also yields a different proof of the *Poisson integral formula* for harmonic functions than the text's, in the current section 2.5, which I may show you later.

Question. Consider $D = D(0;1) \subseteq \mathbb{C}$. What are all possible invertible conformal transformations $f : \bar{D} \to \bar{D}$? In other words, so that f, f^{-1} are each analytic bijections of the closed unit disk.



Step 1 What if we require $f(0) = 0$? Then consider

$$h(z) = \begin{cases} \dfrac{f(z)}{z} & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

Since h is analytic in $\bar{D}$ except at the point $z = 0$ where it is continuous, the modified rectangle lemma and Morera's Theorem prove that h is analytic in the closed disk (i.e. in a slightly larger open disk). The same reasoning applies to $\cancel{\dfrac{1}{h(z)}}\ \tilde{h}(z)$. Use the maximum modulus principle for $h(z)$ and for $\cancel{\dfrac{1}{h(z)}}\ \tilde{h}(z)$ to show that $f(z) = e^{i\theta} z$ are the only conformal diffeomorphisms in this case. Not very many!!!

$$\boxed{\tilde{h}(z) = \begin{cases} \dfrac{f^{-1}(z)}{z} & z \neq 0 \\ (f^{-1})'(0) & z = 0. \end{cases}}$$

$$\left|\frac{f(z)}{z}\right| = \frac{|f(z)|}{1} \leq 1$$

MP: $\max\{|h(z)| \text{ s.t. } z \in \overline{D(0;1)}\} = \max\{|h(z)| \text{ s.t. } |z| = 1\} \leq 1.$

$\Rightarrow \left|\frac{f(z)}{z}\right| \leq 1 \quad (z \neq 0)$

$|f(z)| \leq |z|.$

Same reasoning to $\tilde{h}(z)$

$\Rightarrow |f^{-1}(z)| \leq |z| \quad (z \neq 0)$, $\underbrace{f^{-1}(z)}_{w}$

$\Rightarrow |w| \leq |f(w)| \quad \forall w \in D(0;1).$

$\Rightarrow |f(z)| = |z|$ in disk.

$|h(z)| = \left|\frac{f(z)}{z}\right| \equiv 1$ inside disk

(ii) max princ. $\Rightarrow h(z) = C = e^{i\theta}$, $|C| = 1$.

$h(z) = e^{i\theta}$

$\Rightarrow f(z) = e^{i\theta} z$!!

Step 2 For $z_0 \in D(0; 1)$, consider the *Mobius transformation* (see p. 340, Chapter 5.2; also a first-week homework problem):

① $g'(z) = \dfrac{1\cdot(1+\cancel{\bar{z}_0 z}) - (z_0 - \cancel{z})\bar{z}_0}{(1+\bar{z}_0 z)^2} = \dfrac{1-|z_0|^2}{(1+\bar{z}_0 z)^2}$

notice $|1+\bar{z}_0 z| \geq 1 - |\bar{z}_0 z| \geq 1 - |z_0| > 0$

a) $$g(z) := \frac{z_0 + z}{1 + \bar{z}_0\, z}.$$

$g(0) = z_0$

Show $g(z)$ is conformal in the closed unit disk: ① $g'(z) = \dfrac{1 - |z_0|^2}{(1 + \bar{z}_0 z)^2}$ exists and is non zero in the closed unit disk.

Notice that $g(0) = z_0$. Show that g transforms the ② unit circle to the unit circle, so that by the maximum modulus ③ principle, $|g(z)| < 1 \ \forall\, z \in D(0; 1)$.

b) Denote the Mobius transform g in part (a) by g_{z_0} because the image of the origin is z_0. Solve the equation

(b) solve for z in terms of w

$z_0 + z = w(1 + \bar{z}_0 z)$

$z - w\bar{z}_0 z = w - z_0 = -z_0 + w$

$z(1 - \bar{z}_0 w) = -z_0 + w$

$$② \quad g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z}_0\, z} = w$$

for w to see that the inverse function to $g_{z_0}(z)$ is given by the related Mobius transformation

$$z = g_{-z_0}(w) = \frac{-z_0 + w}{1 - \bar{z}_0 w}.$$

Combining (a), (b) we see that the $g_{z_0}(z)$ are conformal diffeomorphisms of the unit disk.

② $|g(z)|^2 = g\bar{g} = \left(\dfrac{z_0+z}{1+\bar{z}_0 z}\right)\left(\dfrac{\bar{z}_0+\bar{z}}{1+z_0\bar{z}}\right) = \dfrac{|z_0|^2 + |z|^2 + z_0\bar{z} + z\bar{z}_0}{1 + \bar{z}_0 z + z_0\bar{z} + \underbrace{|z_0|^2|z|^2}_{|z_0|^2}} = 1\,!$

If $|z| = 1$.

If $\boxed{|z|=1}$

③ If $|z| < 1$, $|g(z)| < \max\{|g(\zeta)| \text{ s.t. } |\zeta| = 1\} = 1$

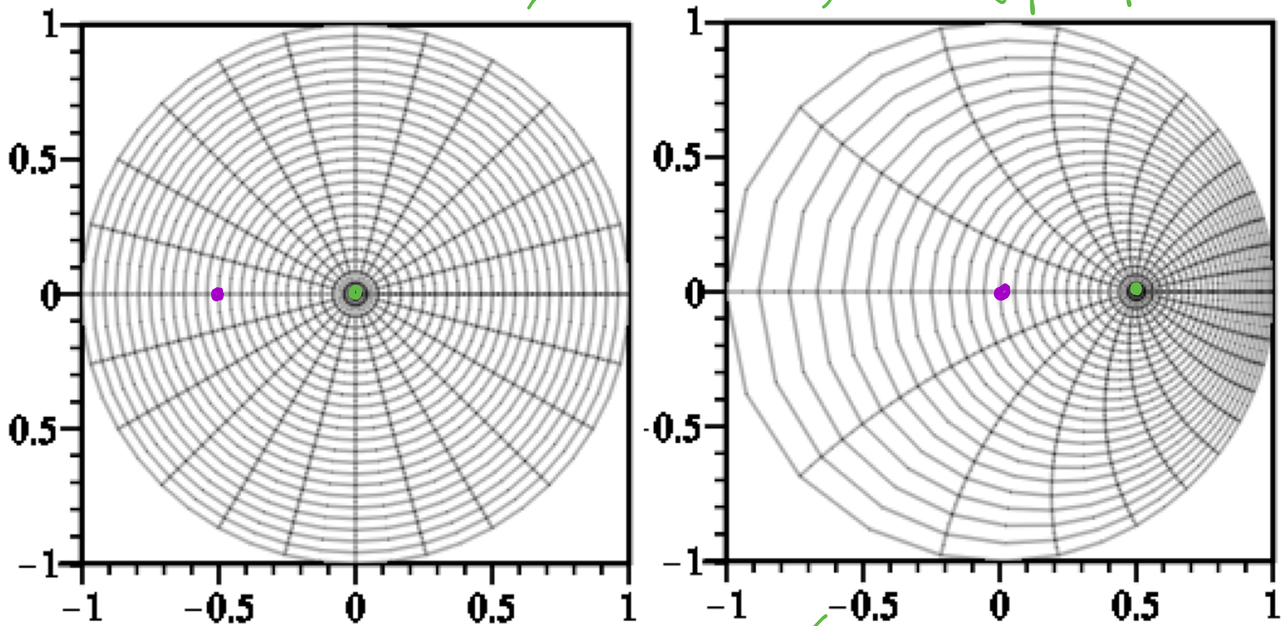
strict.

to be continued!

Here's a Maple picture of how $g_{.5}(z)$ transforms circles concentric to the origin, and rays through the origin. You'll notice that the images of the circles are circles, and the images of the rays are circles (or rays) that hit the unit circle orthogonally. This is not an accident. It turns out that these *Mobius transformations* g_{z_0} are the *isometries* of the *hyperbolic disk*, in *hypberbolic geometry*. (Another circle of ideas for a potential class project.) Notice that $g_{.5}(0) = .5$. Its inverse function is $g_{-.5}(z)$ which maps .5 back to the origin, and maps the origin to $-.5$!

$g_{.5}\, z = \dfrac{.5 + z}{1 + .5 z}$

{lines, circles} $\to$ {lines, circles}
(special for this sort of trans.



$g_{-.5}\, z = \dfrac{-.5 + z}{1 - .5 z}$

Step 3: Combine steps 1 and 2, to show that for $z_0 \in D(0;1)$ *every* conformal diffeomorphism of the unit disk with

$$\underline{f(0) = z_0} \, .$$

can be written as

$$\underline{f(z) = g_{z_0}\left(e^{i\theta} z\right)}$$

for some choice of θ and the Mobius transformations $g_{z_0}(z)$ with $z_0 \in D(0;1)$, from the previous page,

$$g_{z_0}(z) := \frac{z_0 + z}{1 + \bar{z}_0 z} .$$

Not very many!

Consider $g_{z_0}^{-1} \circ f : \overline{D(0,1)} \to \overline{D(0,1)}$

$0 \mapsto z_0 \to 0$.

by step 1,

$$g_{z_0}^{-1}(f(z)) = e^{i\theta} z$$

g_{z_0} : $\Rightarrow$ $f(z) = g_{z_0}\left(e^{i\theta} z\right)$.

Math 4200-001
Week 8 concepts and homework
2.4-2.5
Due Friday October 23 at 11:59 p.m.

2.5 2, 5, 7, 8, 10, 15, 18.
3.1 6, 7. (To get you thinking about sequences and series, for Chapter 3.)

Math 4200
Wednesday October 21
2.5 discuss conformal diffeomorphisms of the disk via the maximum modulus principle, in Monday's notes, and the Poisson integral formula for harmonic functions in the disk, in today's notes. On Friday we will begin Chapter 3 about series representations of analytic functions.

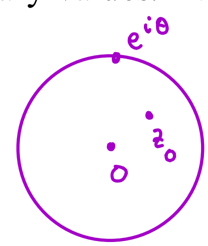
Announcements: Quiz today!

Warm-up exercise:

Application to harmonic function theory (in partial differential equations). There is an analog of the Cauchy integral formula for harmonic functions, that expresses the value of a harmonic function inside a domain in terms of an integral over the boundary which uses the harmonic function's boundary values. It's much messier to write down than the Cauchy integral formula in general - if you wanted to take the real part of the Cauchy integral formula you'd also need to know the boundary values of the conjugate to the harmonic function, to deduce the values of the harmonic function in the interior, so you can't just use the CIF, like we did for the mean value property. In the case where the domain is the unit disk (or a scaled disk), this analog to the CIF is known as the *Poisson integral formula* and we can prove it via the mean value property and Mobius transformations.

Theorem (Poisson integral formula for the unit disk) Let $u \in C^2(\mathrm{D}(0;1)) \cap C(\bar{\mathrm{D}}(0;1))$, and let u be harmonic in $\mathrm{D}(0;1)$. Then the Poisson integral formula recovers the values of u inside the disk, from the boundary values. It may be expressed equivalently in complex form or real form. For $z_0 = x_0 + i\,y_0 = r\,\mathrm{e}^{i\,\varphi}$ with $|z_0| < 1$,

$z_0 = re^{i\varphi}$



- $$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{1-|z_0|^2}{|z_0 - \mathrm{e}^{i\,\theta}|^2}}_{\text{weighting factor}}\, u(\mathrm{e}^{i\,\theta})\,d\theta$$ (if $z_0 = 0$, get MVP)
- $$u(r\cos\varphi, r\sin\varphi) = \frac{1}{2\pi}\int_0^{\pi} \frac{1-r^2}{r^2 - 2\,r\cos(\theta-\varphi)+1}\,u(\cos(\theta), \sin(\theta))\,d\theta$$

* First, check why the CIF formula wouldn't work directly (except for $u(0)$) unless we knew the harmonic conjugate.

$**$ But we do know the mean value property, and we can combine this with the Mobius transformations in yesterday's notes! (Actually we only know the mean value property if u is harmonic on a slightly larger disk than $D(0;1)$, but it also holds for harmonic $u \in C^2(D(0;1)) \cap C(\bar{D}(0;1))$, by a rescaling, limiting process.) In any case, consider the Mobius transformation

$$g_{z_0}(z) = \frac{z_0 + z}{1 + \bar{z}_0 z}$$

Then $u\left(g_{z_0}(z)\right)$ is harmonic on the unit disk (do you remember why, from a Chapter 1 homework problem?). So by the mean value property for the composition,

$$u(z_0) = u\left(g_{z_0}(0)\right) \overset{\text{MVP}}{=} \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$

Now we just change variables, and after some computations out pops the Poisson integral formula! Consider α as a function of θ on the unit circle via

$$\bullet \quad g_{z_0}\left(e^{i\alpha}\right) = e^{i\theta}$$

$$\bullet\bullet \quad g_{-z_0}\left(e^{i\theta}\right) = e^{i\alpha}$$

So,

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$

$$= \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\alpha'(\theta)\, d\theta.$$

To get $\alpha'(\theta)$ we differentiate e.g. the second change of variables formula, using the chain rule for curves and regular Calculus

$$\bullet\bullet \quad \frac{d}{d\theta}\left(g_{-z_0}\left(e^{i\theta}\right)\right) = \frac{d}{d\theta}\left(e^{i\alpha}\right)$$

$$\bullet \quad g_{-z_0}{}'\left(e^{i\theta}\right) i\, e^{i\theta} = i\, e^{i\alpha}\, \alpha'(\theta). \quad \bullet$$

From yesterday's notes, $g_{-z_0}{}'(z) = \dfrac{1 - |z_0|^2}{\left(1 - \bar{z}_0 z\right)^2}$ so the identity above for $\alpha'(\theta)$ reads

$$\frac{1 - |z_0|^2}{\left(1 - \bar{z}_0\, e^{i\theta}\right)^2}\, i\, e^{i\theta} = i\, \frac{-z_0 + e^{i\theta}}{1 - \bar{z}_0\, e^{i\theta}}\, \alpha'(\theta)$$

so (repeating some equations on this page hoping for lecture clarity):

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(g_{z_0}\left(e^{i\alpha}\right)\right) d\alpha$$

$$g_{z_0}\left(e^{i\alpha}\right) = e^{i\theta}, \quad g_{-z_0}\left(e^{i\theta}\right) = e^{i\alpha}$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u\left(e^{i\theta}\right)\alpha'(\theta)\, d\theta$$

where $\alpha'(\theta)$ satisfies the identity

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)^2}\, i\, e^{i\theta} = i\, \frac{-z_0 + e^{i\theta}}{1-\bar{z}_0\, e^{i\theta}}\, \alpha'(\theta)$$

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)}\, \frac{e^{i\theta}}{-z_0 + e^{i\theta}} = \alpha'(\theta)$$

$$\frac{1-|z_0|^2}{\left(1-\bar{z}_0\, e^{i\theta}\right)\left(1-z_0 e^{-i\theta}\right)} = \alpha'(\theta)$$

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z_0|^2}{\left|z_0 - e^{i\theta}\right|^2}\, u\left(e^{i\theta}\right) d\theta \ !$$

QED!!

- Harmonic functions exist and are uniquely determined by their boundary values - we know that from the maximum principle for continous boundary values, and it's even true if the boundary values are only piecewise continuous....in the disk the harmonic functions can be expressed using Fourier series, or with the Poisson integral formula we just proved, and as we've mentioned, they describe various physical phenomena, such as equilibrium temperature distributions in 2-dimensional plates having controlled boundary temperatures....also related to random walk phenomena in probability, other applications.

http://mathfaculty.fullerton.edu/mathews/c2003/DirichletProblemDiskMod.html

Extra Example 1. Find the function $u(x, y) = u(r\cos\theta, r\sin\theta)$ that is harmonic in the unit disk $D_1(0) = \{z : |z| < 1\}$,

and takes on the boundary values $u(\cos\theta, \sin\theta) = U(\theta) = \begin{cases} 1, & \text{for } \frac{\pi}{2} < t < \pi, \\ 0, & \text{for } -\frac{\pi}{2} < t < \frac{\pi}{2}, \\ -1, & \text{for } -\pi < t < -\frac{\pi}{2}. \end{cases}$

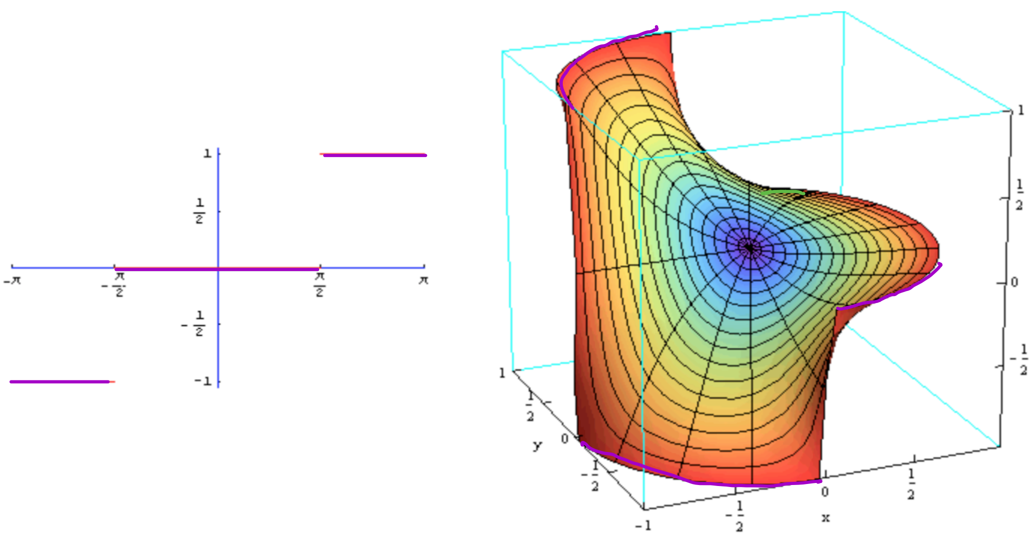


Figure 1. The graphs of $U(\theta) = u(\cos\theta, \sin\theta)$ and $u(x, y) = u(r\cos\theta, r\sin\theta)$.